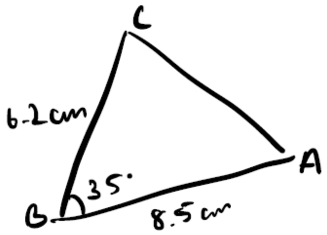
A Level Mathematics A

H240/03 Pure Mathematics and Mechanics

Question Set 5

1. Triangle ABC has $AB = 8.5$ cm, $BC = 6.2$ cm and angle $B = 35°$.

Calculate the area of the triangle. **[2]**



$A = \frac{1}{2} ab\sin C°$

$\frac{1}{2}(6.2)(8.5)\sin 35° = 15.1\text{cm}^2$

2 A sequence of transformations maps the curve $y = e^x$ to the curve $y = e^{2x+3}$.

Give details of these transformations. **[3]**

$y = e^x \qquad y = e^{2x+3}$

- stretch curve horizontally, parallel to x-axis, by scale factor of $\frac{1}{2}$
- transform curve by $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$

3 The functions f and g are defined for all real values of x by

$f(x) = 2x^2 + 6x$ and $g(x) = 3x + 2$.

(a) Find the range of f. **[3]**

$f(x) = 2x^2 + 6x \qquad g(x) = 3x + 2$

$\{ f(x) \in \mathbb{R} : f(x) \geq 0 \}$

(b) Give a reason why f has no inverse. **[1]**

- f(x) is a many-to-one function
- which when inversed, becomes one-to-many relationship
- and can't be simplified to turn into a form of $f^{-1}(x)$

(c) Given that $fg(-2) = g^{-1}(a)$, where a is a constant, determine the value of a. [4]

$f(g(-2))$: $2(3x+2)^2 + 6(3x+2)$

$2(3(-2)+2)^2 + 6(3(-2)+2) = 8$

$g^{-1}(a)$: $g(x) = 3x+2$

$y = 3x+2$

$y-2 = 3x$

$x = \frac{y-2}{3}$

$g^{-1}(x) = \frac{x-2}{3}$

$\frac{a-2}{3} = 8$

$a = 26$

(d) Determine the set of values of x for which $f(x) > g(x)$. Give your answer in set notation. [3]

$f(x) > g(x)$

$2x^2 + 6x > 3x + 2$

$2x^2 + 3x - 2 > 0$

$(2x-1)(x+2) > 0$

$x > \frac{1}{2}$, $x < -2$

$\{x \in \mathbb{R} : x > \frac{1}{2} \text{ or } x < -2\}$

4 A curve has equation $y = 2\ln(k-3x) + x^2 - 3x$, where k is a positive constant.

(a) Given that the curve has a point of inflection where $x = 1$, show that $k = 6$. [5]

$y = 2\ln(k-3x) + x^2 - 3x$

$\frac{dy}{dx} = (-3)\left(2 \frac{1}{k-3x}\right) + 2x - 3$

$= \frac{-6}{k-3x} + 2x - 3$

$\frac{d^2y}{dx^2} = \frac{(k-3x)(0) - (-6)(-3)}{(k-3x)^2} + 2$

$= \frac{-18}{(k-3x)^2} + 2$

$x = 1 \quad \frac{d^2y}{dx^2} = 0$

$$\frac{-18}{(k-3)^2} + 2 = 0$$

$$\frac{-18}{(k-3)^2} = -2$$

$$-18 = -2(k-3)^2$$

$$9 = (k-3)^2$$

$$\pm 3 = k-3$$

$$k = 6 \text{ or } 0$$

$k = 6$

It is also given that the curve intersects the x-axis at exactly one point.

(b) Show by calculation that the x-coordinate of this point lies between 0.5 and 1.5. **[2]**

$2\ln(6-3\times0.5) + 0.5^2 - 3\times0.5 = 1.76 > 0$

$2\ln(6-3\times1.5) + 1.5^2 - 3\times1.5 = -1.44 < 0$

change of sign & function is continuous so x-coordinate lies between 0.5 & 1.5

(c) Use the Newton-Raphson method, with initial value $x_0 = 1$, to find the x-coordinate of the point where the curve intersects the x-axis, giving your answer correct to 5 decimal places. Show the result of each iteration to 6 decimal places. **[3]**

$$x_{n+1} = x_n - \frac{2\ln(6-3x_n) + x_n^2 - 3x_n}{\frac{-6}{6-3x_n} + 2x_n - 3}$$

$x_0 = 1$

$x_1 = 1.065741\ldots$

$x_2 = 1.065675\ldots$

$x_3 = 1.065675\ldots$

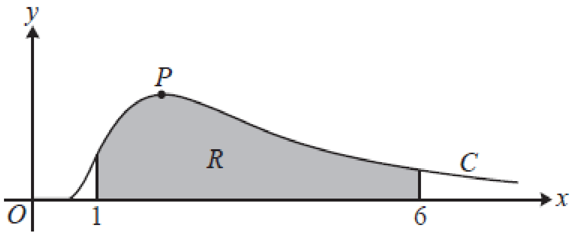
$x_4 = 1.065675\ldots$

$x = 1.06575$

(d) By choosing suitable bounds, verify that your answer to part **(c)** is correct to 5 decimal places. **[1]**

$0.5 < 1.06575 < 1.5$, within range thus correct

5



The diagram shows the curve C with parametric equations

$x = \frac{3}{t}$, $y = t^3 e^{-2t}$, where $t > 0$.

The maximum point on C is denoted by P.

(a) Determine the exact coordinates of P. **[4]**

$x = \frac{3}{t} = 3t^{-1}$ $\qquad y = t^3 e^{-2t}$

$\frac{dx}{dt} = -3t^{-2} = \frac{-3}{t^2}$ $\qquad \frac{dy}{dt} = (t^3)(-2e^{-2t}) + (e^{-2t})(3t^2)$

$= -2t^3 e^{-2t} + 3t^2 e^{-2t} = (3-2t)t^2 e^{-2t}$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = (3-2t)t^2 e^{-2t} \times -\frac{t^2}{3} = -\frac{3-2t}{3} t^4 e^{-2t}$

$\frac{dy}{dx} = 0$ at maximum point $\qquad \frac{3-2t}{3} t^4 e^{-2t} = 0$

$3 - 2t = 0$

$t = \frac{2}{3}$

$x = \frac{3}{\frac{2}{3}} = \frac{9}{2}$ $\qquad y = \left(\frac{2}{3}\right)^3 e^{-2 \times \frac{2}{3}} = \frac{8}{27} e^{-\frac{4}{3}}$

$P\left(\frac{9}{2}, \frac{8}{27} e^{-\frac{4}{3}}\right)$

The shaded region R is enclosed by the curve, the x-axis and the lines $x = 1$ and $x = 6$.

(b) Show that the area of R is given by

$$\int_a^b 3te^{-2t}\,dt,$$

where a and b are constants to be determined. **[3]**

$x = \frac{3}{t} \qquad t = \frac{3}{x} \longrightarrow y = t^3 e^{-2t} \qquad y = \frac{27}{x^3} e^{-\frac{6}{x}}$

$\int_1^6 27x^{-3} e^{-\frac{6}{x}}\,dx = \int_3^{\frac{1}{2}} 27\left(\frac{3}{t}\right)^{-3} e^{-\frac{6}{\frac{3}{t}}} \times -\frac{3}{t^2}\,dt$

$x = \frac{3}{t} \quad \frac{dx}{dt} = -\frac{3}{t^2} \quad = \int_3^{\frac{1}{2}} t^3 e^{-2t} \times -\frac{3}{t^2}\,dt$

$dx = -\frac{3}{t^2}dt \quad = \int_3^{\frac{1}{2}} -3te^{-2t}\,dt$

$1 = \frac{3}{t} \qquad 6 = \frac{3}{t} \qquad = \int_{\frac{1}{2}}^{3} 3te^{-2t}\,dt$

$t = 3 \qquad t = \frac{1}{2}$

(c) Hence determine the exact area of R. **[5]**

$\int_{\frac{1}{2}}^{3} 3te^{-2t}\,dt = \left[(3t)\left(-\frac{1}{2}e^{-2t}\right)\right]_{\frac{1}{2}}^{3} - \int_{\frac{1}{2}}^{3}\left(-\frac{1}{2}e^{-2t}\right)(3)\,dt$

$u = 3t \quad u' = 3$
$v = -\frac{1}{2}e^{-2t} \quad v' = e^{-2t}$

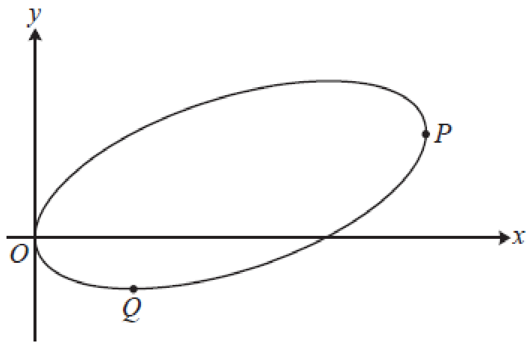
$= \left[-\frac{3}{2}te^{-2t}\right]_{\frac{1}{2}}^{3} + \frac{3}{2}\int_{\frac{1}{2}}^{3} e^{-2t}\,dt$

$= \left[-\frac{3}{2}te^{-2t} + \frac{3}{2} \times -\frac{1}{2}e^{-2t}\right]_{\frac{1}{2}}^{3} = \left[\left(-\frac{3}{2}t - \frac{3}{4}\right)e^{-2t}\right]_{\frac{1}{2}}^{3}$

$= \left(-\frac{3}{2}\times 3 - \frac{3}{4}\right)e^{-2\times 3} - \left(-\frac{3}{2}\times\frac{1}{2} - \frac{3}{4}\right)e^{-2\times\frac{1}{2}} = -\frac{21}{4}e^{-6} + \frac{6}{4}e^{-1}$

$= \frac{(6e^5 - 21)}{4}e^{-6}$

6 **In this question you must show detailed reasoning.**



The diagram shows the curve with equation $4xy = 2(x^2 + 4y^2) - 9x$.

(a) Show that $\frac{dy}{dx} = \frac{4x - 4y - 9}{4x - 16y}$. **[3]**

$$4xy = 2(x^2 + 4y^2) - 9x = 2x^2 - 9x + 8y^2$$

$$(4x)\left(\frac{dy}{dx}\right) + (y)(4) = 4x - 9 + 16y \times \frac{dy}{dx}$$

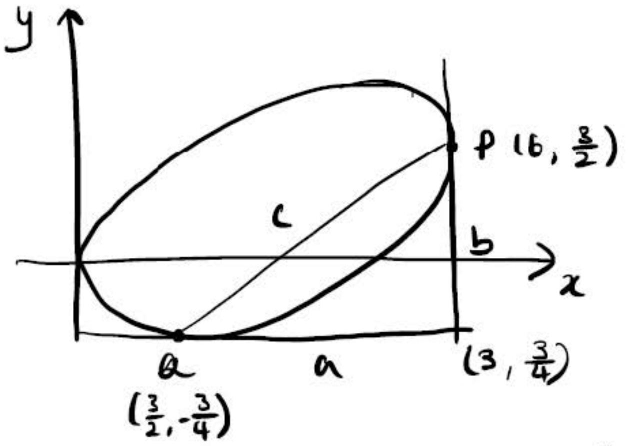
$$4x\frac{dy}{dx} - 16y\frac{dy}{dx} = 4x - 4y - 9$$

$$\frac{dy}{dx}(4x - 16y) = 4x - 4y - 9$$

$$\frac{dy}{dx} = \frac{4x - 4y - 9}{4x - 16y}$$

At the point P on the curve the tangent to the curve is parallel to the y-axis and at the point Q on the curve the tangent to the curve is parallel to the x-axis.

(b) Show that the distance PQ is $k\sqrt{5}$, where k is a rational number to be determined. **[8]**



$$4(4y)y = 2\left((4y)^2+4y^2\right)-9(4y)$$
$$16y^2 = 2\times 20y^2 - 36y$$
$$= 40y^2 - 36y$$
$$24y^2 - 36y = 0$$
$$(2y-3)y = 0$$
$$y = 0 \text{ or } \tfrac{3}{2} \qquad P\left(6, \tfrac{3}{2}\right)$$
$$x = 0 \text{ or } 6$$
$$4x\left(x-\tfrac{9}{4}\right) = 2\left(x^2+4\left(x-\tfrac{9}{4}\right)^2\right)-9x$$
$$4x^2-9x = 2\left(x^2+4\left(x^2-\tfrac{9}{2}x+\tfrac{81}{16}\right)\right)-9x$$
$$4x^2-\cancel{9x} = 2\left(5x^2-18x+\tfrac{81}{4}\right)-\cancel{9x}$$
$$4x^2 = 10x^2-36x+\tfrac{81}{2}$$
$$8x^2 = 20x^2-72x+81$$
$$12x^2-72x+81=0$$
$$4x^2-24x+27=0$$
$$(2x-3)(2x-9)=0$$
$$x = \tfrac{3}{2}, \tfrac{9}{2}$$
$$y = -\tfrac{3}{4}, \tfrac{9}{4} \qquad Q\left(\tfrac{3}{2}, -\tfrac{3}{4}\right)$$

$$PQ = \sqrt{\left(6-\tfrac{3}{2}\right)^2+\left(\tfrac{3}{2}--\tfrac{3}{4}\right)^2}$$
$$= \tfrac{9}{4}\sqrt{5}$$
$$k = \tfrac{9}{4}$$

Total Marks for Question Set 5: 50 Marks