

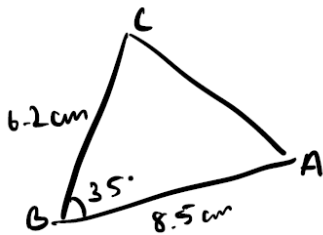
A Level Mathematics A
H240/03 Pure Mathematics and Mechanics

Question Set 5

1. Triangle ABC has $AB = 8.5$ cm, $BC = 6.2$ cm and angle $B = 35^\circ$.

Calculate the area of the triangle.

[2]



$$A = \frac{1}{2} ab \sin C$$

$$\frac{1}{2} (6.2)(8.5) \sin 35^\circ = 15.1 \text{ cm}^2$$

2. A sequence of transformations maps the curve $y = e^x$ to the curve $y = e^{2x+3}$.

Give details of these transformations.

[3]

$$y = e^x \quad y = e^{2x+3}$$

- stretch curve horizontally, parallel to x -axis, by scale factor of $\frac{1}{2}$
- transform curve by $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$

3. The functions f and g are defined for all real values of x by

$$f(x) = 2x^2 + 6x \text{ and } g(x) = 3x + 2.$$

- (a) Find the range of f .

[3]

$$f(x) = 2x^2 + 6x \quad g(x) = 3x + 2$$
$$\{ f(x) \in \mathbb{R} : f(x) \geq 0 \}$$

- (b) Give a reason why f has no inverse.

[1]

- $f(x)$ is a many-to-one function
- which when inversed, becomes one-to-many relationship
- and can't be simplified to turn into a form of $f^{-1}(x)$

(c) Given that $fg(-2) = g^{-1}(a)$, where a is a constant, determine the value of a .

[4]

$$\begin{aligned} f(g(-2)) &: 2(3x+2)^2 + 6(3x+2) \\ & 2(3(-2)+2)^2 + 6(3(-2)+2) = 8 \\ g^{-1}(a) &: g(x) = 3x + 2 & y = 3x + 2 \\ & & y - 2 = 3x \\ & & x = \frac{y-2}{3} \\ g^{-1}(x) &= \frac{x-2}{3} \\ \frac{a-2}{3} &= 8 & a = 26 \end{aligned}$$

(d) Determine the set of values of x for which $f(x) > g(x)$. Give your answer in set notation. [3]

$$\begin{aligned} f(x) &> g(x) \\ 2x^2 + 6x &> 3x + 2 \\ 2x^2 + 3x - 2 &> 0 \\ (2x-1)(x+2) &> 0 \\ x &> \frac{1}{2}, x < -2 \end{aligned}$$
$$\{x \in \mathbb{R} : x > \frac{1}{2} \text{ or } x < -2\}$$

4 A curve has equation $y = 2 \ln(k-3x) + x^2 - 3x$, where k is a positive constant.

(a) Given that the curve has a point of inflection where $x = 1$, show that $k = 6$.

[5]

$$\begin{aligned} y &= 2 \ln(k-3x) + x^2 - 3x \\ \frac{dy}{dx} &= (-3) \left(2 \frac{1}{k-3x} \right) + 2x - 3 \\ &= \frac{-6}{k-3x} + 2x - 3 \\ \frac{d^2y}{dx^2} &= \frac{(k-3x)(0) - (-6)(-3)}{(k-3x)^2} + 2 \\ &= \frac{-18}{(k-3x)^2} + 2 \end{aligned}$$

$$x=1 \quad \frac{d^2y}{dx^2} = 0$$

$$\frac{-18}{(k-3)^2} + 2 = 0$$

$$\frac{-18}{(k-3)^2} = -2$$

$$-18 = -2(k-3)^2$$

$$9 = (k-3)^2$$

$$\pm 3 = k-3$$

$$k = 6 \text{ or } 0$$

$$k = 6$$

It is also given that the curve intersects the x -axis at exactly one point.

(b) Show by calculation that the x -coordinate of this point lies between 0.5 and 1.5. [2]

$$2 \ln(6 - 3 \times 0.5) + 0.5^2 - 3 \times 0.5 = 1.76 > 0$$

$$2 \ln(6 - 3 \times 1.5) + 1.5^2 - 3 \times 1.5 = -1.44 < 0$$

change of sign & function is continuous
so x -coordinate lies between 0.5 & 1.5

(c) Use the Newton-Raphson method, with initial value $x_0 = 1$, to find the x -coordinate of the point where the curve intersects the x -axis, giving your answer correct to 5 decimal places. Show the result of each iteration to 6 decimal places. [3]

$$x_{n+1} = x_n - \frac{2 \ln(6 - 3x_n) + x_n^2 - 3x_n}{\frac{-6}{6 - 3x_n} + 2x_n - 3}$$

$$x_0 = 1$$

$$x_1 = 1.065741 \dots$$

$$x_2 = 1.065675 \dots$$

$$x_3 = 1.065675 \dots$$

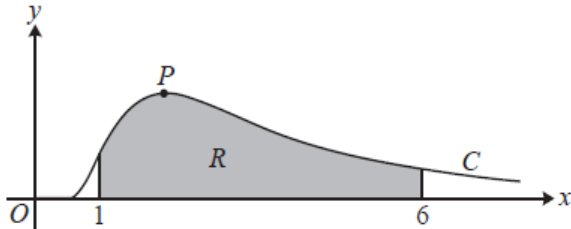
$$x_4 = 1.065675 \dots$$

$$x = 1.06575$$

- (d) By choosing suitable bounds, verify that your answer to part (c) is correct to 5 decimal places. [1]

0.5 < 1.06575 < 1.5, within range thus correct

5



The diagram shows the curve C with parametric equations

$$x = \frac{3}{t}, y = t^3 e^{-2t}, \text{ where } t > 0.$$

The maximum point on C is denoted by P .

- (a) Determine the exact coordinates of P . [4]

$$x = \frac{3}{t} = 3t^{-1} \quad y = t^3 e^{-2t}$$

$$\frac{dx}{dt} = -3t^{-2} = \frac{-3}{t^2} \quad \frac{dy}{dt} = (t^3)(-2e^{-2t}) + (e^{-2t})(3t^2)$$

$$= -2t^3 e^{-2t} + 3t^2 e^{-2t} = (3-2t)t^2 e^{-2t}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = (3-2t)t^2 e^{-2t} \times \frac{t^2}{3} = \frac{3-2t}{3} t^4 e^{-2t}$$

$$\frac{dy}{dx} = 0 \text{ at maximum point} \quad \frac{3-2t}{3} t^4 e^{-2t} = 0$$

$$3 - 2t = 0$$

$$t = \frac{3}{2}$$

$$x = \frac{3}{\frac{3}{2}} = 2 \quad y = \left(\frac{3}{2}\right)^3 e^{-2 \times \frac{3}{2}} = \frac{27}{8} e^{-3}$$

$$P\left(2, \frac{27}{8} e^{-3}\right)$$

The shaded region R is enclosed by the curve, the x -axis and the lines $x = 1$ and $x = 6$.

(b) Show that the area of R is given by

$$\int_a^b 3te^{-2t} dt,$$

where a and b are constants to be determined.

[3]

$$x = \frac{3}{t} \quad t = \frac{3}{x} \rightarrow y = t^3 e^{-2t} \quad y = \frac{27}{x^3} e^{-\frac{6}{x}}$$

$$\int_1^6 27x^{-3} e^{-\frac{6}{x}} dx = \int_{\frac{1}{3}}^{\frac{1}{2}} 27 \left(\frac{3}{t}\right)^{-3} e^{-\frac{6}{\frac{3}{t}}} x^{-\frac{3}{t^2}} dt$$

$$x = \frac{3}{t} \quad \frac{dx}{dt} = -\frac{3}{t^2} = \int_{\frac{1}{3}}^{\frac{1}{2}} t^3 e^{-2t} x - \frac{3}{t^2} dt$$

$$dx = -\frac{3}{t^2} dt = \int_{\frac{1}{3}}^{\frac{1}{2}} -3te^{-2t} dt$$

$$1 = \frac{3}{t}$$

$$6 = \frac{3}{t}$$

$$= \int_{\frac{1}{2}}^{\frac{1}{3}} 3te^{-2t} dt$$

$$t = 3$$

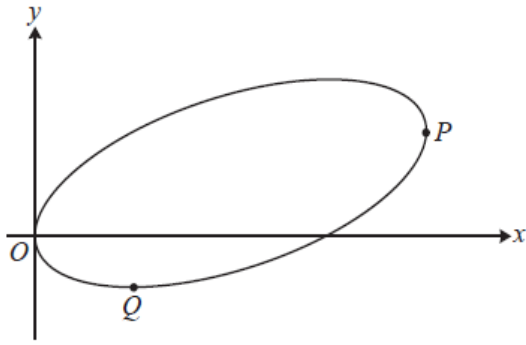
$$t = \frac{1}{2}$$

(c) Hence determine the exact area of R .

[5]

$$\begin{aligned} \int_{\frac{1}{2}}^{\frac{1}{3}} 3te^{-2t} dt &= \left[(3t) \left(-\frac{1}{2} e^{-2t}\right) \right]_{\frac{1}{2}}^{\frac{1}{3}} - \int_{\frac{1}{2}}^{\frac{1}{3}} \left(-\frac{1}{2} e^{-2t}\right) (3) dt \\ u=3t \quad u' &= 3 \\ v = -\frac{1}{2} e^{-2t} \quad v' &= e^{-2t} \\ &= \left[-\frac{3}{2} t e^{-2t} \right]_{\frac{1}{2}}^{\frac{1}{3}} + \frac{3}{2} \int_{\frac{1}{2}}^{\frac{1}{3}} e^{-2t} dt \\ &= \left[-\frac{3}{2} t e^{-2t} + \frac{3}{2} x - \frac{1}{2} e^{-2t} \right]_{\frac{1}{2}}^{\frac{1}{3}} = \left[\left(-\frac{3}{2} t - \frac{3}{4}\right) e^{-2t} \right]_{\frac{1}{2}}^{\frac{1}{3}} \\ &= \left(-\frac{3}{2} \times 3 - \frac{3}{4}\right) e^{-2 \times 3} - \left(-\frac{3}{2} \times \frac{1}{2} - \frac{3}{4}\right) e^{-2 \times \frac{1}{2}} = -\frac{21}{4} e^{-6} + \frac{6}{4} e^{-1} \\ &= \frac{(6e^5 - 21)}{4} e^{-6} \end{aligned}$$

6 In this question you must show detailed reasoning.



The diagram shows the curve with equation $4xy = 2(x^2 + 4y^2) - 9x$.

(a) Show that $\frac{dy}{dx} = \frac{4x - 4y - 9}{4x - 16y}$.

[3]

$$4xy = 2(x^2 + 4y^2) - 9x = 2x^2 - 9x + 8y^2$$
$$(4x)\left(\frac{dy}{dx}\right) + (y)(4) = 4x - 9 + 16y \times \frac{dy}{dx}$$

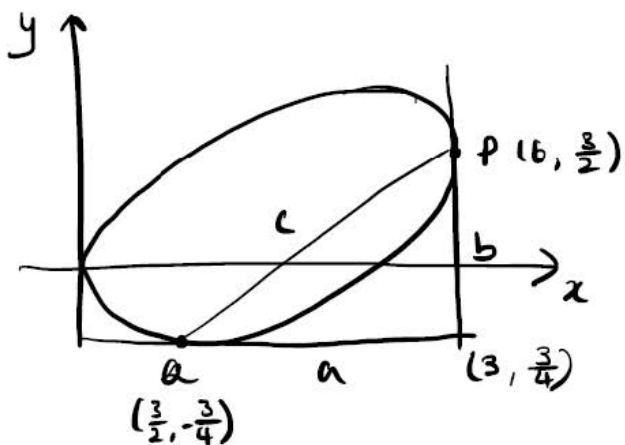
$$4x \frac{dy}{dx} - 16y \frac{dy}{dx} = 4x - 4y - 9$$

$$\frac{dy}{dx} (4x - 16y) = 4x - 4y - 9$$

$$\frac{dy}{dx} = \frac{4x - 4y - 9}{4x - 16y}$$

At the point P on the curve the tangent to the curve is parallel to the y -axis and at the point Q on the curve the tangent to the curve is parallel to the x -axis.

(b) Show that the distance PQ is $k\sqrt{5}$, where k is a rational number to be determined. [8]



$$4(4y)y = 2((4y)^2 + 4y^2) - 9(4y)$$

$$16y^2 = 2 \times 20y^2 - 36y$$

$$= 40y^2 - 36y$$

$$24y^2 - 36y = 0$$

$$(2y - 3)y = 0$$

$$y = 0 \text{ or } \frac{3}{2}$$

$$x = 0 \text{ or } b$$

$$P(6, \frac{3}{2})$$

$$PQ = \sqrt{(6 - \frac{3}{2})^2 + (\frac{3}{2} - -\frac{3}{4})^2}$$

$$= \frac{9}{4}\sqrt{5}$$

$$k = \frac{9}{4}$$

$$4x(x - \frac{9}{4}) = 2(x^2 + 4(x - \frac{9}{4})^2) - 9x$$

$$4x^2 - 9x = 2(x^2 + 4(x^2 - \frac{9}{2}x + \frac{81}{16})) - 9x$$

$$4x^2 - 9x = 2(5x^2 - 18x + \frac{81}{4}) - 9x$$

$$4x^2 = 10x^2 - 36x + \frac{81}{2}$$

$$8x^2 = 20x^2 - 72x + 81$$

$$12x^2 - 72x + 81 = 0$$

$$4x^2 - 24x + 27 = 0$$

$$(2x - 3)(2x - 9) = 0$$

$$x = \frac{3}{2}, \frac{9}{2}$$

$$y = -\frac{3}{4}, \frac{9}{4}$$

$$Q(\frac{3}{2}, -\frac{3}{4})$$

Total Marks for Question Set 5: 50 Marks

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